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Optimal Layout of Chord Graph into the Windmill Graph

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Abstract— Graph embedding is an important and extensively studied theory in parallel computing. A great deal of research has been devoted to finding "good" embedding of one network into another. The embedding of a guest graph G into a host graph H is defined by a pair of injective functions between them. The edge congestion of an embedding is the maximum number of edges of the guest graph that are embedded on any single edge e of the host graph. The optimal layout problem deals with finding the embedding for which the sum of all the shortest paths in H corresponding to the edges in G is minimum. In this paper, we find the optimal layout of embedding the chord graph into the windmill graph.

Keywords— Embedding, chord graph, optimal set, windmill graph, layout.

I. INTRODUCTION

Graph embedding has gained importance in studying the computational capabilities of interconnection and virtual networks in recent years. Graph embedding is an important technique that maps a guest graph into a host graph, usually an interconnection network [10]. A graph embedding is an ordered pair $\prec f, P_f \succ$ of one-to-one functions that maps a guest graph $G = (V(G), E(G))$into a host graph$H = (V(H), E(H))$, where f is a map from V(G) to V(H)and P_f assigns to each edge (u, v) of G, a shortest path $P_f(u, v)$ in H [6]. Figure 1 gives a simple illustration of the embedding of graph into a path. The edge congestion of an embedding $EC_{\prec f, P_f \succ}(e)$ is the maximum number of edges of the guest graph that are embedded on any single edge e of the host graph[7].The optimal layout problem deals with finding the embedding for which the sum of all the shortest paths in H corresponding to the edges in G is minimum [8]. The layout of an embedding$\prec f, P_f \succ$from G to H is defined as $L_{\prec f, P_f \succ}(G, H) = \sum_{e \epsilon E H} EC_f(e)$. The optimal layout is given by [2]

$$L(G, H) = \min_{\prec f, P_f \succ} L_{\prec f, P_f \succ}(G, H).$$

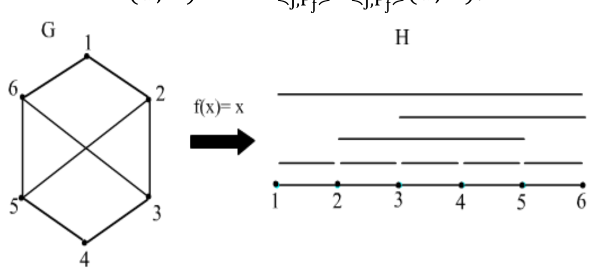

Figure 1: Embedding of graph into Path.

In addition to parallel computing the layout problem also finds application in VLSI circuit design, cloning and structural engineering. In this paper, we find the optimal layout of embedding the chord graph into the windmill graph. The following result plays an important role in optimal layout computation.

Maximum Induced Subgraph Problem: [4] The maximum induced subgraph problem plays an important role in the wirelength problem. The problem is to find a subset of vertices of a given graph such that the number of edges in the subgraph induced by this subset is maximal among all induced subgraphs with the same number of vertices. Mathematically, for a given m, if $I_G(m) = \max_{A \subseteq V, |A|=m} |I_G(A)|$ where $I_G(A) = \{(u, v) \in E: u, v \in A\}$, then the problem is to find a set $A \subseteq V$such that$|A| = m$ and $I_G(m) = |I_G(A)|$. such a set A is called an optimal set [2]. The optimal set helps in proving that the obtained layout is minimum.

Lemma: (k–Partition Wirelength Lemma):[3] Let G be an r – regular graph and $\prec f, P_f \succ$ be an embedding of G into a graph H. Let $E^k(H)$ denote a collection of edge of H with each edge in H repeated exactly k times. Let $\{S_1, S_2, S_3, .., S_p\}$ be a partition of $E^k(H)$such that each $S_i$is an edge cut of H. For $1 \le i \le p$, the removal of edges of $S_i$splits H into 2 components $H_i^1$and H_i^2. Let $G_i^1 = G[F^{-1}(H_i^1)]$ and $G_i^2 = G[F^{-1}(H_i^2)]$. Let $EC_{\prec f, P_f \succ}$ denotes the sum of edge congestion over all the edges in S_i. $S_i$satisfies the following conditions for $EC_{\prec f, P_f \succ}$ to be minimum.

1. For every edge $(a,b) \in G_i^j, j = 1,2,\ P_f(a,b)$ has no edges in S_i and for every $(a,b) \in G$ with $a \in G_i^1$ and $b \in G_i^2,\ P_f(a,b)$ has exactly one edge in S_i.
2. Either $V(G_i^1)$ or $V(G_i^2)$ is an optimal vertex s. Then, $EC_{<f, P_f>}(S_i) = \sum_{e \in S_i} EC_{<f, P_f>}(e) = r|V(G_i^1)| - 2|E(G_i^1)|$ and $L_{<f, P_f>}(G,H) = \frac{1}{k}\sum_{i=1}^{p} EC_{<f, P_f>}(S_i)$.

In this paper we obtain the optimal layout of a circulant variant known as the chord graph into a windmill graph. The rest of the paper is organized as follows. Section II gives some fundamental definitions and preliminaries. In section III the optimal layout of embedding chord graph into windmill graph is obtained. In section IV we conclude the paper.

II. METHODOLOGY

Definition 1: For $i = 1,2,3$, let K_{t_i} be a complete graph on 2^{r_i} vertices each whose cardinality is predefined as follows:

i) $t_1 = 2^{r_1} = t_2$, $r_1 \le n-2$

ii) $t_3 = 2^{r_3}$, where $r_3 \le n-1$

The resultant graph union $\uplus K_{t_i}$ is obtained by joining two adjacent vertices of K_{t_2} and K_{t_3} each with any vertex $v \in K_{t_1}$ and is called windmill graph, denoted by $WM(K_1, K_2, K_3)$. See Figure 2.

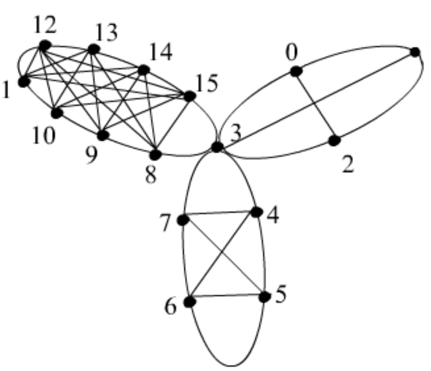

Figure2: Windmill graph

Definition 2: [9] A chord graph CH_n with 2^n vertices has vertex set $V(CH_n) = \{0,1,\dots,2^n-1\}$. Edge $e = (i,j) \in E(CH_n)$ if and only if $i + 2^k = mod_{2^n} j$ or $j + 2^k = mod_{2^n} i$, for all $k \in \{0,1,\dots,n-1\}$. See Figure3.The chord graph CH_n is a $(2n-1)$ regular graph.

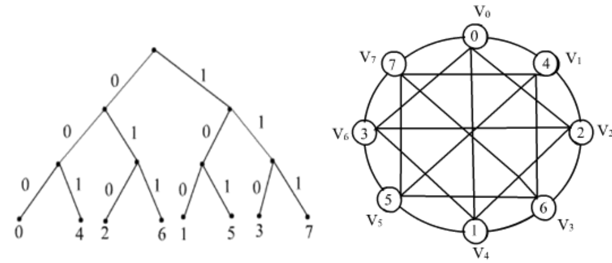

Figure 3: (a) Chord labeling prelude (b)Chord graph

Definition 3:[9] A subchord of CH_n is its subgraph induced by i, $1 \le i \le 2^n$ vertices. Two subchords CH_{k_1} and CH_{k_2} on k_1 and k_2 nodes respectively are said to be strong neighboring subchords, if each vertex of CH_{k_1} (or) CH_{k_2} has exactly two neighbors in CH_{k_2}(or) CH_{k_1}.

Definition 4:[9] Let $l = \sum_{j=1}^{p} 2^{c_j}, 0 \le c_1 < c_2 < \cdots < c_p$ be the base 2 representation for any positive integer $l < 2^n$.If $S \subseteq V(CH_n)$is a subset on l vertices made up of a disjoint union of c_j-subchords, $1 \le j \le p$ such that each c_i-subchord lies in the strong neighbourhood of every c_j-subchord for $j > i$, then the subchord induced by S is called an IChord.

Remark 1:[9] If CH_n is an IChord and $|S|$ is even, then $|E(CH_n[S])| = \sum_{j=1}^{p}(p-1)2^{c_j+1} + 2^{c_j-1}(2c_j - 1)$ and if $|S|$ is odd, then $|E(CH_n[S])| = \sum_{j=2}^{p}(p-1)2^{c_j+1} + 2^{c_j-1}(2c_j - 1)$.

Theorem:[1] For $1 \le i \le 2^n L_i = \{0,1,2,\dots,(i-1)\}$ is an optimal set inCH_n.

Remark 2: By the construction of CH_n,we make an observation wherein, if the consecutive vertices of the optimal set L_{2^n} are decomposed into 2^{n-i} disjoint sets with each set carrying 2^i vertices, the subgraph induced by every one of the these sets is isomorphic to $CH_n[L_{2^i}]$ and hence optimal.

Proposition 1: [5] Let G be a symmetric vertices transitive graph on n vertices.Then $G\backslash v$ is a maximum subgraph on $n-1$ vertices for any $v \in V(G)$.

Lemma 1: For$i = 1,2,3$,$WM_i = \{(i-1)2^{r_1},\dots,k\}$, where

$$K = \begin{cases} 2^{r_i} - 2 & when\ i = 1 \\ 2^{r_{i-1}} + 2^{r_i} - 1 & when\ i = 2 \\ 2^{r_{i-2}} + 2^{r_{i-1}} + 2^{r_i} - 1 & when\ i = 3 \end{cases}$$

is an optimal set in CH_n.

Proof: From the construction of CH_n, for $i \in \{2,3\}$, the subgraph induced by each WM_i is an Ichord on 2^{r_i} vertices and hence by Remark 2 it is optimal. Since $CH_n[2^{r_1}]$ is symmetric and in turn vertex transitive, by Proposition 1, the induced subgraph $CH_n[WM_1] = CH_n[2^{r_1}]\backslash 2^{r_1}$)is an optimal set in CH_n.

Chord Graph Labeling: Consider a complete binary tree of height r. Label the left child of every internal vertex of the tree as 0 and the right child as 1. Concatenate the labels along the tree starting from a leaf to the root and label the leaf vertex with the decimal representation of the obtained binary number. with the Gather the labels of the leaf vertices of the complete binary tree from left to right and label the outer cycle of the chord graph in the clockwise sense in that order[4]. See Figure 3(b).

Windmill Graph Labeling: Label the vertices of $WM(K_1, K_2, K_3)$ using vertices in the order $\prec 0,1,\ldots,2^{n-1} \succ$ starting from K_{t_1} and label $V(K_{t_1}) - \{v\}$using the labels$\prec 0,1,\ldots,2^{r_1}-2 \succ$in clockwise direction. For i=2,3 label $V(K_{t_i}) - \{v\}$ using $\prec 2^{r_1}, 2^{r_1}+1,\ldots,2^{r_1}+2^{r_2}-1 \succ$ and$\prec 2^{r_1}, 2^{r_2},\ldots,2^{n}-1 \succ$respectively in clockwise direction. Figure 2 illustrates this labeling.

Embedding: Define an embedding $\prec f, P_f \succ$from CH_n to $WM(K_1, K_2, K_3)$ such that $f(x) = x$ and $P_f(x,y)$is any shortest path between $f(x)$and$f(y)$ for $(x,y) \in E(CH_n)$.

III. RESULTS AND DISCUSSION

Lemma 2: The embedding $\prec f, P_f \succ$ defined in Section II induces the minimum layout.

Proof: For $i = 1,2,\ldots,2^n - 1$, let S_i^1 be the edge cut such that the removal of edges in the cut splits windmill graph $WM(K_1, K_2, K_3)$into two components A_i and $\overline{A_i}$ where

$$V(A_i) = \begin{cases} i-1 : i \in \{1,2,\ldots,2^{r_1}-1\} \\ \quad i \quad : i \in \{2^{r_1},\ldots,2^n-1\} \end{cases}$$

Let $G_i$be the graph induced by $\{f^{-1}(u) : u \in V(A_i)\}$.Since the vertex set is made up of isolated vertices,$V(G_i)$ is optimal.

For $i = 1,2,3$, let S_i be the edge cut whose removal splits $WM(K_1, K_2, K_3)$ into two components B_i and $\overline{B_i}$ where $V(B_i) = WM_i$. Let G_{B_i}be the graph induced by $\{f^{-1}(u) : u \in WM_i\}$. By Lemma1,$V(G_{B_i})$is optimal in CH_n. All edge cuts are illustrated in Figure 4. The two cuts satisfy the conditions (i)-(ii) of the k-Partition Wirelength Lemma.

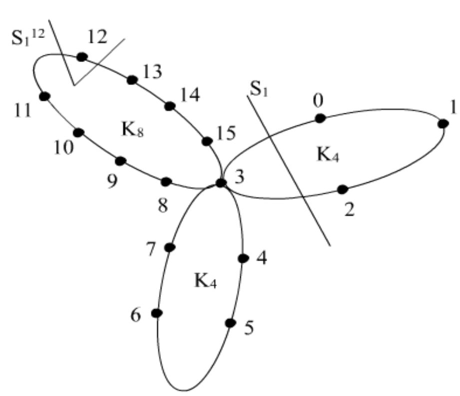

Figure 4: Edge cuts of $WM(K_4, K_4, K_8)$

In addition the sets $\{S_i^1 : i = 1,2,\ldots,2^n - 1\} \cup \{S_i : i = 1,2,3\}$ form a 2-Partition Wirelength Lemma. Hence $L(CH_n, WM(K_1, K_2, K_3)) = L_{\prec f, P_f \succ}(CH_n, WM(K_1, K_2, K_3))$.

Theorem: The optimal layout of embedding chord graph CH_ninto windmill graph $WM(K_1, K_2, K_3)$ is given by $L(CH_n, WM(K_1, K_2, K_3)) = \frac{1}{2}\{(2n-1)\{(2^n-1) + \sum_{i=1}^{3}|V(K_{t_i})| - 1\} - 2\sum_{i=1}^{3}[(2r_i 1)(2^{r_i-1}-1)]$.

Proof:

For $i = 1\ to 2^n - 1$, $EC_{\prec f, P_f \succ}(S_i^1) = (2n-1)(1) - 2(0) = 2n - 1$.

For $i = 1,2,3, EC_{\prec f, P_f \succ}(S_i) = (2n-1)(|V(K_{t_i})| - 1) - 2(|E(CH_n[K_{t_i}])|)$.

Then,

$$L(CH_n, WM(K_1, K_2, K_3)) = \frac{1}{2}\{\sum_{i=1}^{2^n-1} EC(S_i^1) + \sum_{i=1}^{3} EC(S_i)\}$$

$$= \frac{1}{2}\{(2^n-1)(2n-1)\} + \sum_{i=1}^{3}\{(2n-1)(|V(K_{t_i})| - 1) - 2(2r_i - 1)(2(2^{r_i-1}-1))\}$$

$$= \frac{1}{2}\{(2n-1)[(2^n-1) + [\sum_{i=1}^{3}|V(K_{t_i})| - 1]]\} - 2\{(2r_i-1)(2^{r_i-1}-1)\}$$

IV. CONCLUSION

In this paper we optimized the layout of embedding the chord graph into windmill graph using edge partition techniques and the k-Partition Wirelength Lemma.

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