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Embedding of Circulant Networks Into Cycle-of-butterfly

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Abstract

Circulant networks is one of most popular interconnection networks since it has a simple structure and is easy to implement. Graph embedding is an important parameter to evaluate the quality of an interconnection network and wirelength is an important measure of an embedding. In this paper, we embed circulant network into cycle-of-butterfly with minimum wirelength.

Key Words: Embedding; Congestion; Wirelength; Circulant networks; cycle-of-butterfly.

1 Introduction

Interconnection networks provide an effective mechanism for exchanging data between processors in a parallel computing system. An interconnection network is often represented as a graph, where nodes and edges correspond to processors and communication links between processors, respectively. In the design and analysis of an interconnection network, its graph embedding ability is a major concern. An ideal interconnection network (host graph) is expected to possess excellent graph embedding ability which helps efficiently execute parallel algorithms with regular task graphs (guest graphs) on this network [1].
An embedding of a guest graph G into a host graph H is a one-to-one mapping of the vertex set of G into that of H. The dilation of an embedding is defined as the maximum distance between a pair of vertices of H that are images of adjacent vertices of G. The study of graph embeddings is an important topic in the theory of parallel computation: the existence of such an embedding demonstrates the ability of a parallel computer, whose interconnection network is represented by the host graph, to simulate a parallel algorithm, whose communication structure is described by the guest graph. The dilation can then serve as one of natural measures of the communication delay [2].

The wirelength of a graph embedding arises from VLSI designs, data structures and data representations, networks for parallel computer systems, biological models that deal with cloning and visual stimuli, parallel architecture, structural engineering and so on [9, 10].
Graph embeddings have been well studied for meshes into crossed cubes [11], binary trees into paths [10], binary trees into hypercubes [2, 12], complete binary trees into hypercubes [13], incomplete hypercube in books [14], tori and grids into twisted cubes [15], meshes into locally twisted cubes [16], meshes into faulty crossed cubes [1], meshes into crossed cubes [11], generalized ladders into hypercubes [17], grids into grids [18], binary trees into grids [19], hypercubes into cycles [20, 21], star graph into path [22], snarks into torus [23], generalized wheels into arbitrary trees [24], hypercubes into grids [25], m-sequencial k-ary trees into hypercubes [26], meshes into mobius cubes [27],

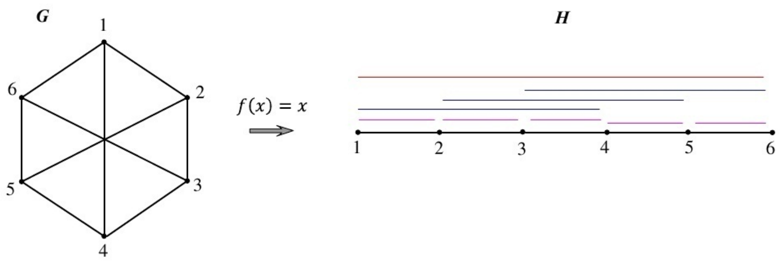

Figure 1: Wiring diagram of Circulant graph G into a path H with $WL_f(G, H) = 19$.

ternary tree into hypercube [28], enhanced and augmented hypercube into complete binary tree [29], circulant into arbitrary trees, cycles, certain multicyclic graphs and ladders [30], hypercubes into cylinders, snakes and caterpillars [31], hypercubes into necklace, windmill and snake graphs [19], embedding of special classes of circulant networks, hypercubes and generalized Petersen graphs [33].
Even though there are numerous results and discussions on the wirelength problem, most of them deal with only approximate results and the estimation of lower bounds [20, 34]. The embeddings discussed in this paper produce exact wirelength.

2 Preliminaries

In this section we give the basic definitions and preliminaries related to embedding problems.

Definition 2.1. (See [34].) *Let G and H be finite graphs with n vertices. An embedding f of G into H is defined as follows:*

1. *f is a bijective map from $V(G) \to V(H)$*
2. *f is a one-to-one map from $E(G)$ to $\{P_f(u,v) : P_f(u,v)$ is a path in H between $f(u)$ and $f(v)$ for $(u,v) \in E(G)\}$.*

Definition 2.2. (See [34].) *The edge congestion of an embedding f of G into H is the maximum number of edges of the graph G that are embedded on any single edge of H. Let $EC_f(G, H(e))$ denote the number of edges (u,v) of G such that e is in the path $P_f(u,v)$ between $f(u)$ and $f(v)$ in H. In other words,*

$$EC_f(G, H(e)) = |\{(u,v) \in E(G) : e \in P_f(u,v)\}|$$

where $P_f(u,v)$ denotes the path between $f(u)$ and $f(v)$ in H with respect to f.

If we think of G as representing the wiring diagram of an electronic circuit, with the vertices representing components and the edges representing wires connecting them, then the edge congestion $EC(G,H)$ is the minimum, over all embeddings $f : V(G) \to V(H)$, of the maximum number of wires that cross any edge of H [36].

Definition 2.3. (See [25].) *The wirelength of an embedding f of G into H is given by*

$$WL_f(G,H) = \sum_{(u,v)\in E(G)} d_H(f(u), f(v)) = \sum_{e\in E(H)} EC_f(G, H(e))$$

where $d_H(f(u), f(v))$ denotes the length of the path $P_f(u, v)$ in H. See Figure 1. Then, the wirelength of G into H is defined as

$$WL(G, H) = \min WL_f(G, H)$$

where the minimum is taken over all embeddings f of G into H.

The wirelength problem [19, 20, 24, 25, 34, 37], of a graph G into H is to find an embedding of G into H that induces the minimum wirelength $WL(G, H)$.

The following two versions of the edge isoperimetric problem of a graph $G(V, E)$ have been considered in the literature [37], and are NP-complete [40].

Problem 1 : Find a subset of vertices of a given graph, such that the edge cut separating this subset from its complement has minimal size among all subsets of the same cardinality. Mathematically, for a given m, if $\theta_G(m) = \min_{A\subseteq V,\ |A|=m} |\theta_G(A)|$ where $\theta_G(A) = \{(u, v) \in E : u \in A, v \notin A\}$, then the problem is to find $A \subseteq V$ such that $|A| = m$ and $\theta_G(m) = |\theta_G(A)|$.

Problem 2 : Find a subset of vertices of a given graph, such that the number of edges in the subgraph induced by this subset is maximal among all induced subgraphs with the same number of vertices. Mathematically, for a given m, if $I_G(m) = \max_{A\subseteq V,\ |A|=m} |I_G(A)|$ where $I_G(A) = \{(u, v) \in E : u, v \in A\}$, then the problem is to find $A \subseteq V$ such that $|A| = m$ and $I_G(m) = |I_G(A)|$.

For a given m, where $m = 1, 2, ..., n$, we consider the problem of finding a subset A of vertices of G such that $|A| = m$ and $|\theta_G(A)| = \theta_G(m)$. Such subsets are called optimal. We say that optimal subsets are nested if there exists a total order $\mathcal{O}$ on the set V such that for any $m = 1, 2, ..., n$, the first m vertices in this order is an optimal subset. In this case we call the order $\mathcal{O}$ an optimal order [37, 38]. This implies that $WL(G, P_n) = \sum_{m=0}^{n} \theta_G(m)$.

Further, if a subset of vertices is optimal with respect to Problem 1, then its complement is also an optimal set. But, it is not true for Problem 2 in general. However for regular graphs a subset of vertices S is optimal with respect to Problem 1 if and only if S is optimal for Problem 2 [37]. In the literature, Problem 2 is defined as the maximum subgraph problem.

Notation. $EC_f(G, H(e))$ will be represented by $EC_f(e)$. For any set S of edges of H, $EC_f(S) = \sum_{e\epsilon S} EC_f(e)$.

Lemma 2.4. (Congestion Lemma) (See [25].) *Let G be an r-regular graph and f be an embedding of G into H. Let S be an edge cut of H such that the removal of edges of S leaves H into 2 components H_1 and H_2 and let $G_1 = f^{-1}(H_1)$ and $G_2 = f^{-1}(H_2)$. Also S satisfies the following conditions:*

(i) *For every edge $(a, b) \in G_i$, $i = 1, 2, P_f(a, b)$ has no edges in S.*

(ii) *For every edge (a, b) in G with $a \in G_1$ and $b \in G_2$, $P_f(a, b)$ has exactly one edge in S.*

(iii) *G_1 is a maximum subgraph on k vertices where $k = |V(G_1)|$.*

Then $EC_f(S)$ is minimum,that is, $EC_f(S) \leq EC_g(S)$ for any other embedding g of G into H, and $EC_f(S) = rk - 2|E(G_1)| = \theta_G(k)$.

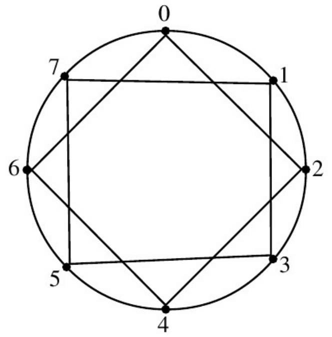

Figure 2: Circulant graph $G(8;\pm\{1,2\})$.

Lemma 2.5. (Partition Lemma) (See [25].) *Let $f : G \to H$ be an embedding. Let $\{S_1, S_2, \ldots, S_p\}$ be a partition of $E(H)$ such that each S_i is an edge cut of H. Then*

$$WL_f(G,H) = \sum_{i=1}^{p} EC_f(S_i).$$

Lemma 2.6. (2-Partition Lemma) [30] *Let $f : G \to H$ be an embedding. Let $[2E(H)]$ denote a collection of edges of H repeated exactly 2 times. Let $\{S_1, S_2, ..., S_m\}$ be a partition of $[2E(H)]$ such that each S_i is an edge cut of H. Then*

$$WL_f(G,H) = \frac{1}{2}\sum_{i=1}^{m} EC_f(S_i).$$

3 Circulant networks

The circulant is a natural generalization of the double loop network and was first considered by Wong and Coppersmith [39]. Circulant graphs have been used for decades in the design of computer and telecommunication networks due to their optimal fault-tolerance and routing capabilities [4]. It is also used in VLSI design and distributed computation [5, 6, 7]. The term circulant comes from the nature of its adjacency matrix. A matrix is circulant if all its rows are periodic rotations of the first one. Circulant matrices have been employed for designing binary codes[8]. Theoretical properties of circulant graphs have been studied extensively and surveyed in [5]. Every circulant graph is a vertex transitive graph and a Cayley graph [9]. Most of the earlier research concentrated on using the circulant graphs to build interconnection networks for distributed and parallel systems [4, 5].

Definition 3.1. (See [9].) *A circulant undirected graph $G(n;\pm S)$, where $S \subseteq \{1, 2, \ldots, \lfloor n/2 \rfloor\}$, $n \geq 3$ is defined as a graph consisting of the vertex set $V = \{0, 1, \ldots, n-1\}$ and the edge set $E = \{(i,j) : |j-i| \equiv s(mod\ n),\ s \in S\}$.*

The circulant graph shown in Figure 2 is $G(8;\pm\{1,2\})$. It is clear that $G(n;\pm 1)$ is the undirected cycle C_n and $G(n;\pm\{1, 2, \ldots, \lfloor n/2 \rfloor\})$ is the complete graph K_n. Further $G(n;\pm\{1, 2, \ldots, \lfloor j \rfloor\})$, $1 \leq j < \lfloor n/2 \rfloor$, $n \geq 3$ is a $2j$-regular graph.

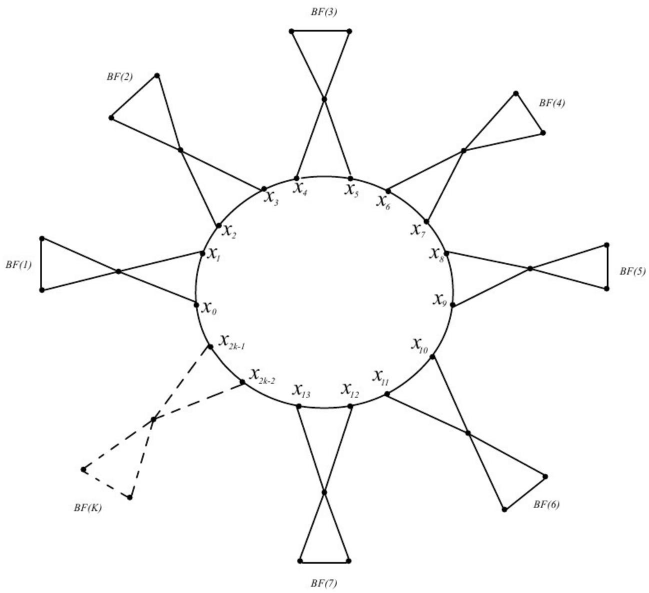

Figure 3: Cycle-of-butterfly.

Theorem 3.2. (See [30].) *A set of k consecutive vertices of $G(n;\pm 1)$, $1 \le k \le n$ induces a maximum subgraph of $G(n;\pm S)$, where $S = \{1, 2, \ldots, j\}$, $1 \le j < \lfloor n/2 \rfloor$, $n \ge 3$.*

Theorem 3.3. (See [30].) *The number of edges in a maximum subgraph on k vertices of $G(n;\pm S)$, $S = \{1, 2, \ldots, j\}$, $1 \le j < \lfloor n/2 \rfloor$, $1 \le k \le n$, $n \ge 3$ is given by*

$$\xi = \begin{cases} k(k-1)/2 & ; \; k \le j+1 \\ kj - j(j+1)/2 & ; \; j+1 < k \le n-j \\ \frac{1}{2}\{(n-k)^2 + (4j+1)k - (2j+1)n\} & ; \; n-j < k \le n. \end{cases}$$

4 Wirelength of circulant networks into cycle-of-butterfly

In this section, we compute the exact wirelength of circulant networks into cycle-of-butterfly.

Definition 4.1. (See [35].) *A cycle-of-butterfly is a graph unified by a bone cycle BC and k butterfly $B(1), B(2), \ldots, B(k)$ with $BR(1), BR(2), \ldots, BR(k)$ as the bottom rungs, respectively, such that each $BR(i)$ is contained in the BC where $1 \le i \le k$.*

The structure of a cycle-of-butterfly graph is shown in Fig 3, where $(x_0, x_1, \ldots, x_{2k-1})$ are the BR(1), BR(2),. . . , BR(k) respectively. Clearly this type of cycle-of-butterfly contains $5k$ vertices and denoted by $COB(k)$. Clearly this type of cycle-of-butterfly contains $5k$ vertices and it is denoted by $COB(k)$.

Embedding Algorithm

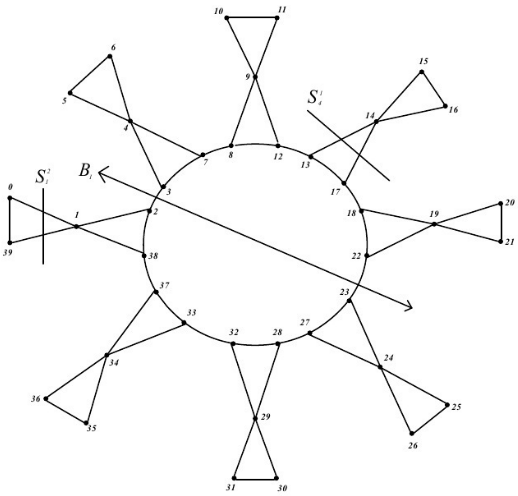

Figure 4: The edge cuts of $COB(8)$.

Input : A circulant network $G(n; \pm\{1, 2, \ldots, j\})$, $1 \leq j < \lfloor n/2 \rfloor$ and a cycle-of-butterfly $COB(k)$, where $n = 5k$.

Algorithm : Label the consecutive vertices of $G(n; \pm 1)$ in $G(n; \pm\{1, 2, \ldots, j\})$, as $0, 1, \ldots, n-1$ in the clockwise sense. Label the vertices of $COB(k)$ as follows: Label BC(1)as $5k-2$, 2 and for $2 \leq i \leq k$, label BC(i) as $5i-7$, $5i-3$. Label B(1) as $5k-1$, 0, 1 and for $2 \leq i \leq k$ B(i) as $5i-6, 5i-5, 5i-4$.

Output : An embedding f of $G(n; \pm\{1, 2, \ldots, j\})$ into $COB(k)$ given by $f(x) = x$ with minimum wirelength.

Proof of correctness : We assume that the labels represent the vertices to which they are assigned.

Case 1 (**k even):**
For $1 \leq i \leq \frac{k}{2}$, let $S_i = S_i^{'}$ be the set of edges which contains all the edges of $B(i)$ and $B(i+\frac{k}{2})$, along with the edges of BC(i) and BC($i+\frac{k}{2}$). For $1 \leq j \leq \frac{k}{2}$, let B_j be the set of edges which contains the edge between $B(i)$ and $B(i+1)$, along with the edge between $B(\frac{k}{2}+i)$ and $B(\frac{k}{2}+i+1)_{mod\ k}$. For $1 \leq i \leq k$, let S_i^1 be the set of edges between $B(i)$ and BC(i), let S_i^2 be the set of edges incident to top vertex of $B(i)$. See Figure 4. Then $\{S_i, S_i^{'} : 1 \leq i \leq \frac{k}{2}\} \cup \{B_j : 1 \leq j \leq \frac{k}{2}\} \cup \{S_i^1, S_i^2 : 1 \leq i \leq k\}$ is a partition of $[2E(COT(k))]$.

For $1 \leq i \leq \frac{k}{2}$, $E(COB(k)) \setminus S_i$ has two components H_{i1} and H_{i2} where

$$V(H_{i1}) = \{5i-3, 5i-2, \ldots, 5i-3+2k\}.$$

Let $G_{i1} = f^{-1}(H_{i1})$ and $G_{i2} = f^{-1}(H_{i2})$. By Theorem 3.3, G_{i1}, is an optimal set, and each S_i satisfies conditions (i), (ii) and (iii) of the congestion lemma. Therefore $EC_f(S_i)$ is minimum. Similarly $EC_f(S_i^{'})$ is minimum.

For $1 \leq j \leq \frac{k}{2}$, $E(COB(k)) \setminus B_j$ has two components H_{j1} and H_{j2} where

$$V(H_{j1}) = \{5j-2, 5j-1, \ldots, 5j+1+2k\}.$$

Let $G_{j1} = f^{-1}(H_{j1})$ and $G_{j2} = f^{-1}(H_{j2})$. By Theorem 3.3, G_{j1}, is an optimal set, and each B_j satisfies conditions (i), (ii) and (iii) of the congestion lemma. Therefore $EC_f(B_j)$ is minimum.

For $1 \leq i \leq k$, $E(COB(k))\setminus S_i^1$ has two components H_{i1}^1 and H_{i2}^1 where

$$V(H_{i1}^1) = \begin{cases} \{0, 1, 5k-1\} & ; \; if \quad i = 0 \\ \{5i-6, 5i-5, 5i-4\} & ; \; if \quad i \neq 0 \end{cases}$$

Let $G_{i1}^1 = f^{-1}(H_{i1}^1)$ and $G_{i2}^1 = f^{-1}(H_{i2}^1)$. Since G_{i1}^1, is an optimal set, each S_i^1 satisfies conditions (i), (ii) and (iii) of the congestion lemma. Therefore $EC_f(S_i^1)$ is minimum.

For $1 \leq i \leq k$, $E(COB(k))\setminus S_i^2$ has two components H_{i1}^2 and H_{i2}^2 where

$$V(H_{i1}^2) = \{5i-5, 5i-4\}$$

Let $G_{i1}^2 = f^{-1}(H_{i1}^2)$ and $G_{i2}^2 = f^{-1}(H_{i2}^2)$. Since G_{i1}^2, is an optimal set, each S_i^2 satisfies conditions (i), (ii) and (iii) of the congestion lemma. Therefore $EC_f(S_i^2)$ is minimum. The 2-Partition Lemma implies that the wirelength is minimum.

Case 2 (k Odd):

For $1 \leq i \leq \frac{k}{2}$, when i is odd, let $S_i = S_i^{'}$ be the set of edges which contains all the edges of $B(i)$ along with the edge BC(i) and the edge between BC$(i+\frac{k}{2})$ and BC$(i+\frac{k}{2}+1)$, when i is even, let $S_i = S_i^{'}$ be the set of edges which contains edge of BC(i) along with the edge of B$(i+\frac{k}{2})$ and BC$(i+\frac{k}{2})$. For $1 \leq i \leq k$, let S_i^1 be the set of edges between $B(i)$ and BC(i), let S_i^2 be the set of edges incident to top vertex of $T(i)$. Then $\{S_i, S_i^{'} : 1 \leq i \leq \frac{k}{2}\} \cup \{S_i^1, S_i^2 : 1 \leq i \leq k\}$ is a partition of $[2E(COB(k))]$.

As in Case 1, it is easy to prove that the wirelength is minimum.

Theorem 4.2. *The exact wirelength of circulant graph $G(n; \pm\{1, 2, \ldots, j\})$, $1 \leq j < \lfloor n/2 \rfloor$ into $COB(k)$ is given by*

$$WL(G, COB(k)) = \begin{cases} \frac{k}{2}\{\theta_G(\frac{5k}{2} - 3) + \frac{1}{2}\theta_G(\frac{5k}{2}) + \theta_G(3) + \theta_G(2)\} & ; \; if \quad k \quad even \\ \frac{k}{2}\{\theta_G(\frac{5k-3}{2}) + \theta_G(3) + \theta_G(2)\} & ; \; if \quad k \quad odd \end{cases}$$

Proof: Following the notations of the Embedding Algorithm, we divide the proof into two cases.

Case 1 (k even):

By congestion lemma,

i) $EC_f(S_i) = \theta_G((\frac{5k}{2}) - 3)$, $1 \leq i \leq \frac{k}{2}$

ii) $EC_f(S_i^{'}) = \theta_G((\frac{5k}{2}) - 3)$, $1 \leq i \leq \frac{k}{2}$

iii) $EC_f(B_j) = \theta_G(\frac{5k}{2})$, $1 \leq j \leq \frac{k}{2}$

iv) $EC_f(S_i^1) = \theta_G(3)$, *and* $EC_f(S_i^2) = \theta_G(2)$, $1 \leq i \leq k$.

Then by 2-partition lemma,

$$
\begin{aligned}
WL(G, COB(k)) &= \frac{1}{2}\Bigg\{\sum_{i=1}^{k/2} EC_f(S_i) + \sum_{i=1}^{k/2} EC_f(S_i^{'}) + \sum_{j=1}^{k/2} EC_f(B_j) \\
&+ \sum_{i=1}^{k} \left[EC_f(S_i^1) + EC_f(S_i^2)\right] \Bigg\} \\
&= \frac{1}{2}\Bigg\{\sum_{i=1}^{k/2} \theta_G((\frac{5k}{2}) - 3) + \sum_{i=1}^{k/2} \theta_G((\frac{5k}{2}) - 3) + \sum_{j=1}^{k/2} \theta_G(\frac{5k}{2}) \\
&+ k\theta_G(3) + k\theta_G(2)\Bigg\} \\
&= \frac{1}{2}\Bigg\{\frac{k}{2}\theta_G((\frac{5k}{2}) - 3) + \frac{k}{2}\theta_G((\frac{5k}{2}) - 3) + \frac{k}{2}\theta_G(\frac{5k}{2}) \\
&+ k\theta_G(3) + k\theta_G(2)\Bigg\} \\
&= \frac{1}{2}\left\{k\theta_G((\frac{5k}{2}) - 3) + \frac{k}{2}\theta_G(\frac{5k}{2}) + k\theta_G(3) + k\theta_G(2)\right\} \\
&= \frac{k}{2}\left\{\theta_G(\frac{5k}{2} - 3) + \frac{1}{2}\theta_G(\frac{5k}{2}) + \theta_G(3) + \theta_G(2)\right\}
\end{aligned}
$$

Case 2 (k Odd):
By congestion lemma,

i) $EC_f(S_i) = \theta_G\left(\lfloor \frac{5k}{2} \rfloor\right)$, $1 \leq i \leq \frac{k}{2}$

ii) $EC_f(S_i^{'}) = \theta_G\left(\lfloor \frac{5k}{2} \rfloor\right)$, $1 \leq i \leq \frac{k}{2}$

iii) $EC_f(S_i^1) = \theta_G(3)$, *and* $EC_f(S_i^2) = \theta_G(2)$, $1 \leq i \leq k$.

Then by 2-partition lemma,

$$
WL(G, COB(k)) = \frac{1}{2}\left\{\sum_{i=1}^{k/2} EC_f(S_i) + \sum_{i=1}^{k/2} EC_f(S_i^{'}) + \sum_{i=1}^{k} [EC_f(S_i^1) + EC_f(S_i^2)]\right\}
$$

$$
= \frac{1}{2}\left\{\sum_{i=1}^{k/2} \theta_G\left(\left\lfloor \frac{5k}{2} \right\rfloor\right) + \sum_{i=1}^{k/2} \theta_G \left\lfloor \frac{5k}{2} \right\rfloor + k\theta_G(3) + k\theta_G(2)\right\}
$$

$$
= \frac{k}{2}\left\{\theta_G(\frac{5k-3}{2}) + \theta_G(3) + \theta_G(2)\right\}.
$$

Hence the proof.

5 Conclusion

In this paper, we embed circulant networks into cycle-of-butterfly to yield the minimum wirelength. In our opinion, the reduction technique developed in this paper is very powerful and may be applied to the fault-tolerant embeddings of cycle-of-butterfly in other kinds of interconnection networks.

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