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A Study of Nonlinear Vibration of Euler-Bernoulli Beams Using Coupling Between The Aboodh Transform And The Homotopy Perturbation Method

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Abstract— Aboodh transform (AT) in combination with the homotopy perturbation method (HPM) is employed to solve the nonlinear differential equation of motion for Euler-Bernoulli beams in a unified way. Aboodh transform based homotopy perturbation method (ATHPM) is found to give analytic solutions with all perturbative corrections to both the displacement and the oscillation frequency in a very simple and straight forward manner. Here, we have also demonstrated the sophistication and simplicity of this technique.

Keywords— Aboodh Transform, Homotopy Perturbation Method, Euler-Bernoulli Beam

I. INTRODUCTION

From simply supported to clamped-clamped structures, understanding of the dynamic response of large amplitudes of beam vibration has attracted numerous researchers and engineers over several decades. They are encountered in many engineering applications from bridges to robot arms. The main inquisitiveness is to find the difference in the characteristics of dynamic response from those defined via linear theory. Researchers have explored the problem of beam vibration with different boundary conditions and hypotheses. They predicted the nonlinear frequency of beams which is principal to the design of engineering structures. The Euler-Bernoulli theory of beams, relating the beam's deflection and applied load, provides a reasonable explanation for the bending behavior of long isotropic beams. These vibration problems of the beam can be solved by both analytical and numerical methods.

The solution to the nonlinear problems are difficult to find and most of them are not exactly solvable. Although the numerical solution to the nonlinear problems is easy, one desires to find the analytical solution to get a better insight of the problem. There are many techniques for solving nonlinear problems such as the harmonic balance method, Krylov-Bogoliubov-Mitropolsky method, weighted linearization method, perturbation procedure for limit cycle analysis, modified Lindstedt-Poincare method, Adomain decomposition method, artificial parameter method, and Nikiforov-Uvarov method [1-9]. Not only these methods have complex calculations, but they fail to handle problems with strong nonlinearity.

The homotopy perturbation method (HPM) has been found to be very efficient for solving nonlinear equations with known initial or boundary values especially for systems with strong non-linearity in classical and quantum mechanical problems [10-14]. In this method, the solution is given in an infinite series usually converges to an accurate solution [13]. Aboodh introduced a transform derived from the classical Fourier integral for solving ordinary and partial differential equations easily in the time (t) domain [15]. Aboodh transform (AT) has been applied to different types of problems and is found to be very simple but powerful technique [17,18].

In this article, we have applied ATHPM to solve the nonlinear differential equations with different types of Euler-Bernoulli beams in a generalized way to obtain the approximate displacement u and the oscillating frequency ω with high accuracy.

This paper is organized as follows. In section II, we demonstrate briefly the formulation of ATHPM. Applications of ATHPM to the Euler-Bernoulli beams problems have been shown in section III. In section IV we present some examples cases. Finally, in section V we provide a brief discussion and our conclusion.

II. FORMULATION OF ATHPM

Aboodh transform is defined for function of exponential order in a set A where

$$A=\{u(t):\exists M,k_1,k_2>0,|u(t)|<Me^{-vt}\} \tag{1}$$

The constant M must be finite number and k_1,k_2 may be finite or infinite. If $u(t)$ is the piecewise continuous function for $t\geq 0$, the corresponding Aboodh Transform $u(v)$ is defined as

$$A[u(t)]=u(v)=\frac{1}{v}\int_0^{\infty}u(t)e^{-vt}dt,t\geq 0,k_1\leq v\leq k_2 \tag{2}$$

Some properties of Aboodh Transform which are necessary for our calculations are

$$A[u'(t)]=vu(v)-\frac{u(0)}{v}-u(0) \tag{3}$$

$$A[u''(t)]=v^2u(v)-\frac{u'(0)}{v}-u(0) \tag{4}$$

$$A[\cos at]=\frac{1}{v^2+a^2},A[t\sin at]=\frac{2a}{(v^2+a^2)^2} \tag{5}$$

$$A[t^n]=\frac{n!}{v^{n+2}} \tag{6}$$

Let us consider a nonlinear non-homogeneous differential equation as

$$Lu(t)+\omega^2u(t)+Ru(t)+Nu(t)=g(t) \tag{7}$$

with the initial condition $u(0)=u_0(0)$ and $u_0'(0)=0$. Here, L is the second order linear differential operator $(L\equiv\partial^2/\partial t^2)$, R is the linear operator having an order less than L , N is the nonlinear operator, $g(t)$ is the non-homogeneous term and ω^2 is any parameter.

Now, taking the Aboodh Transform on both sides of (7) we get

$$A[Lu(t)]+\omega^2A[u(t)]+A[Ru(t)]+A[Nu(t)]=A[g(t)] \tag{8}$$

Using the differential properties of the Aboodh Transform as mentioned above and the initial conditions (8) can be written

$$u(v)=\left(\frac{1}{v^2+\omega^2}\right)u(0)+\frac{u'(0)}{v(v^2+\omega^2)}-\left\{\left(\frac{1}{v^2+\omega^2}\right)A[Ru(t)]+\left(\frac{1}{v^2+\omega^2}\right)A[Nu(t)]+\left(\frac{1}{v^2+\omega^2}\right)A[g(t)]\right\} \tag{9}$$

Taking Inverse Aboodh Transform on both sides of (9) leads to

$$u(t)=U_0(t)-\left\{A^{-1}\left[\left(\frac{1}{v^2+\omega^2}\right)A[Ru(t)]\right]+A^{-1}\left[\left(\frac{1}{v^2+\omega^2}\right)A[Nu(t)]\right]+A^{-1}\left[\left(\frac{1}{v^2+\omega^2}\right)A[g(t)]\right]\right. \tag{10}$$

where the solution of zeroth order correction is

$$U_0(t)=\left(\frac{1}{v^2+\omega^2}\right)u(0)+\frac{u'(0)}{v(v^2+\omega^2)} \tag{11}$$

According to the homotopy perturbation method, we can write

$$u(t)=\sum_{n=0}^{\infty}p^nu_n(t) \tag{12}$$

and the nonlinear term as

$$Nu(t)=\sum_{n=0}^{\infty}p^nH_n(u) \tag{13}$$

where He's polynomial $H_n(u)$ is given by

$$H_n(u)=\frac{1}{n!}\frac{d^n}{dp^n}\left[N\left(\sum_{n=0}^{\infty}p^nu_n(t)\right)\right]_{p=0},n=0,1,2,3\ldots \tag{14}$$

Here, n corresponds to the order of correction terms. Substituting the value of $u(t)$ and $Nu(t)$ in (10), we get

$$\sum_{n=0}^{\infty}p^nu_n(t)=U_0(t)-p\left\{A^{-1}\left[\left(\frac{1}{v^2+\omega^2}\right)A[R\sum_{n=0}^{\infty}p^nu_n(t)\right]+A^{-1}\left[\left(\frac{1}{v^2+\omega^2}\right)A\left[\sum_{n=0}^{\infty}p^nH_n(t)\right]\right]+A^{-1}\left[\left(\frac{1}{v^2+\omega^2}\right)A[g(t)]\right]\right\} \tag{15}$$

Equation (15) is the combination of the Aboodh Transform and the homotopy perturbation method using He's polynomials. Comparing the coefficient of like power of p, we get from (15) the following equations

$$p^0 : u_0(t) = U_0(t) \tag{16}$$

$$p^1 : u_1(t) = -\left\{ A^{-1}\left[\left(\frac{1}{v^2+\omega^2}\right)A[Ru_0(t)]\right] + A^{-1}\left[\left(\frac{1}{v^2+\omega^2}\right)A[H_0(u_0(t))]\right]\right] + A^{-1}\left[\left(\frac{1}{v^2+\omega^2}\right)A[g(t)]\right] \tag{17}$$

$$p^2 : u_2(t) = -\left\{ A^{-1}\left[\left(\frac{1}{v^2+\omega^2}\right)A[Ru_1(t)]\right] + A^{-1}\left[\left(\frac{1}{v^2+\omega^2}\right)A[H_1(u_0(t),u_1(t)]\right]\right\} \tag{18}$$

The approximate solution is

$$u(t) = \lim_{p\to 1}\sum_{n=0}^{\infty} p^n u_n(t) = u_0(t) + u_1(t) + u_2(t) + \ldots \tag{19}$$

III. Application of ATHPM to Solve Nonlinear Differential Equations

To assess the advantages and accuracy of ATHPM, we consider the following non-linear differential equation as

$$\frac{d^2u}{dt^2} + A_1u + A_2u^2 + A_3u^3 + A_5u^5 = 0 \tag{20}$$

Now, for our purpose we rewrite (20) as

$$\frac{d^2u}{dt^2} + \omega^2 u = \left(\omega^2 - A_1\right)u - A_2u^2 - A_3u^3 - A_5u^5 \tag{21}$$

where, ω is the unknown frequency to be determined. Using the differential properties of the Aboodh Transform and the initial condition $u(0) = a, u'(0) = 0$ (21) can be written as

$$u(v) = \left(\frac{1}{v^2+\omega^2}\right)a + (\omega^2 - A_1)\left(\frac{1}{v^2+\omega^2}\right)A[u] - A_2\left(\frac{1}{v^2+\omega^2}\right)A[u^2] - A_3\left(\frac{1}{v^2+\omega^2}\right)A[u^3] - A_5\left(\frac{1}{v^2+\omega^2}\right)A[u^5] \tag{22}$$

Taking the inverse Aboodh Transform on both sides of (22) we get

$$u(t) = a\cos\omega t + (\omega^2 - A_1)A^{-1}\left[\left(\frac{1}{v^2+\omega^2}\right)A[u]\right] + A_2A^{-1}\left[\left(\frac{1}{v^2+\omega^2}\right)A[u^2]\right] - A_3A^{-1}\left[\left(\frac{1}{v^2+\omega^2}\right)A[u^3]\right] - A_5A^{-1}\left[\left(\frac{1}{v^2+\omega^2}\right)A[u^5]\right] \tag{23}$$

Applying the HPM to (23) as before, we get the zeroth order correction of the displacement as

$$p^0 : u_0(t) = a\cos\omega t \tag{24}$$

and the first order correction as

$$p^1 : u_1(t) = a\left(\omega^2 - A_1\right)A^{-1}\left[\left(\frac{1}{v^2+\omega^2}\right)A[\cos\omega t]\right] - A_2a^2A^{-1}\left[\left(\frac{1}{v^2+\omega^2}\right)A\left[\cos^2\omega t\right]\right] - A_3a^3A^{-1}\left[\left(\frac{1}{v^2+\omega^2}\right)A\left[\cos^3\omega t\right]\right] - A_5a^5A^{-1}\left[\left(\frac{1}{v^2+\omega^2}\right)A\left[\cos^5\omega t\right]\right] \tag{25}$$

After some mathematical calculation of inverse Aboodh Transform, we get the first order correction as

$$p^1 : u_1(t) = a\cos\omega t - \frac{A_2a^2}{2\omega^2}\left(1 - \cos\omega t\right) - \frac{A_2a^2}{6\omega^2}\left(\cos\omega t - \cos 2\omega t\right) - \frac{1}{8\omega^2}\left(\frac{A_3a^3}{4} + \frac{5A_5a^5}{16}\right)\left(\cos\omega t - \cos 3\omega t\right) - \frac{A_5a^5}{384\omega^2}\left(\cos\omega t - \cos 5\omega t\right) \tag{26}$$

Finally, we obtain the approximate solution by ATHPM up to first order correction as

$$u(t) = -\frac{A_2a^2}{2\omega^2} + \left(a + \frac{A_2a^2}{3\omega^2} - \frac{A_3a^3}{32\omega^2} - \frac{A_5a^5}{384\omega^2}\right)\cos\omega t + \frac{A_2a^2}{6\omega^2}\cos 2\omega t + \frac{1}{8\omega^2}\left(\frac{A_3a^3}{4} + \frac{5A_5a^5}{16}\right)\cos 3\omega t + \frac{A_5a^5}{384\omega^2}\cos 5\omega t \tag{27}$$

and the frequency of oscillation as,

$$\omega = \sqrt{A_1 + \frac{3A_3 a^2}{4} + \frac{5A_5 a^4}{8}} \tag{28}$$

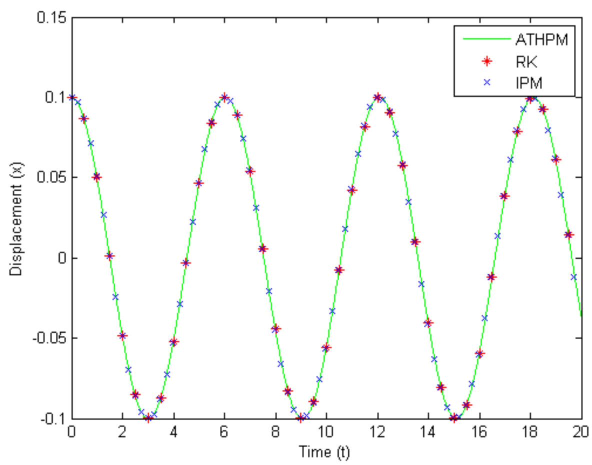

Figure 1. Time(t) vs displacement(x) curves obtained from numerical (RK), ATHMP and IPM with $A_1 = 1$, $A_2 = 0$, $A_3 = 10$, $A_5 = 100$ and a = 0.1

We have plotted the displacement x(t) from numerical solution for $A_1 = 1$, $A_2 = 0$, $A_3 = 10$, $A_5 = 100$, a = 0.1 and compared the same obtained from Runge-Kutta (RK) and Iteration Perturbation Method (IPM) calculations. It is found that the displacement obtained from RK, ATHPM and IPM are matching very closely.

IV. Results and Discussion

Here we discuss different cases of Euler-Bernoulli beams subjected to different kinds of loading.

Case 1

Using ATHPM we obtain the analytical expression for geometrically non-linear vibration of clamped-clamped Euler-Bernoulli beams fixed at one end. This type of geometric nonlinearity arises from nonlinear strain-displacement relationships. Sources of this type of nonlinearity include mid-plane stretching, large curvatures of structural elements and large rotation of elements.

Consider a straight beam on an elastic foundation with length *L*, cross-section area *A*, mass per unit length μ, moment of inertia *I* and modulus of elasticity *E* that is subjected to an axial force of magnitude *F'* as shown in figure 2.

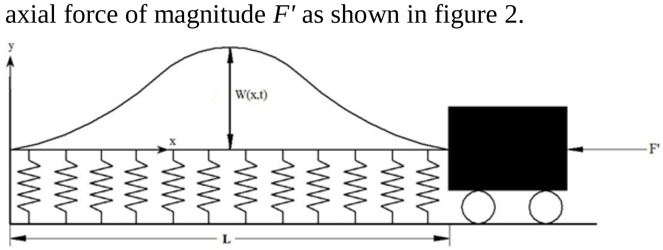

Figure 2. Schematic of Euler-Bernoulli beam subjected to an axial load

We assumed that the cross-sectional area of the beam is uniform and its material is homogenous. Here the schematic is modelled according to the Euler-Bernoulli beam theory [19]. Planes of the cross section remain plane after deformation, straight lines normal to mid plane of the beam remain normal, and straight lines in the transverse direction of the cross section do not change length. First, we assume that there is no in plane deformation. Second, we ignore the transverse shear strains and consequently the rotation of the cross section is due to bending. Lastly, we assume that there are no transverse normal strains. The equation of motion with the effects of mid-plane stretching is given by

$$\mu \frac{\partial^2 W'}{\partial t'^2} + EI \frac{\partial^4 W'}{\partial x'^4} + F' \frac{\partial^2 W'}{\partial x'^2} + C \frac{\partial W'}{\partial t'} + K'W' - \frac{EA}{2L} \frac{\partial^2 W'}{\partial x'^2} \int_0^L \left(\frac{\partial W'}{\partial x'} \right)^2 dx' = U(x',t') \tag{29}$$

where, *C* is the viscous damping coefficient, *K'* is the foundation modulus and *U* is the distributed load in the transverse direction. We assume that here the non-conservative force $U(x',t')$ is zero. So, (29) can be written as

$$\mu \frac{\partial^2 W'}{\partial t'^2} + EI \frac{\partial^4 W'}{\partial x'^4} + F' \frac{\partial^2 W'}{\partial x'^2} + C \frac{\partial W'}{\partial t'} + K'W' - \frac{EA}{2L} \frac{\partial^2 W'}{\partial x'^2} \int_0^L \left(\frac{\partial W'}{\partial x'} \right) dx' = 0 \tag{30}$$

For convenience, we have used the following non-dimensional variables

$$x = \frac{x'}{L}, W = \frac{W'}{R}, F = \frac{F'L^2}{EI}, K = \frac{K'L^4}{EI}, t = t'\sqrt{\frac{EI}{\mu L^4}} \tag{31}$$

where, $R = \sqrt{I/A}$ is the radius of gyration of the cross-section. With the help of (31), (30) can be written as

$$\frac{\partial^2 W}{\partial t^2} + EI \frac{\partial^4 W}{\partial x^4} + F \frac{\partial^2 W}{\partial x^2} + C \frac{\partial W}{\partial t} + KW - \frac{EA}{2L} \frac{\partial^2 W}{\partial x^2} \int_0^L \left(\frac{\partial W}{\partial x} \right)^2 dx = 0 \tag{32}$$

Using separation of variables as $W(x,t) = \psi(x)u(t)$, where $\psi(x)$ is the first eigen mode of the beam and applying the Galerkin method, the equation of motion is obtained from (32) as

$$\frac{d^2u(t)}{dt^2} + A_1 u + A_3 u^3 = 0 \qquad (33)$$

where, $A_1 = a_1 + a_2 F + K, A_3 = a_3$ and a_1, a_2, a_3 are given as

$$a_1 = \frac{\int_0^1 \frac{d^4\psi(x)}{dx^4}\psi(x)dx}{\int_0^1 \psi^2(x)dx}, a_2 = \frac{\int_0^1 \frac{d^2\psi(x)}{dx^2}\psi(x)dx}{\int_0^1 \psi^2(x)dx}$$
$$a_3 = -\frac{1}{2}\frac{\int_0^1 \left(\frac{d^2\psi(x)}{dx^2}\int_0^1 \left(\frac{d\psi(x)}{dx}\right)^2 dx\right)\psi(x)dx}{\int_0^1 \psi^2(x)dx} \qquad (34)$$

Equation (33) is the governing nonlinear vibration equation of Euler-Bernoulli beams. The center of the beam is subjected to the following initial conditions as $u(0) = a, u'(0) = 0$. Here a is the non-dimensional maximum amplitude of oscillation. Comparing (20) and (33), we get the approximate solution up to first order from (27) as

$$u(t) = \left(a - \frac{A_3 a^3}{32\omega^2}\right)\cos\omega t + \left(\frac{A_3 a^3}{32\omega^2}\right)\cos 3\omega t \qquad (35)$$

and the frequency of oscillation is obtained from (28) as

$$\omega = \sqrt{A_1 + \frac{3A_3 a^3}{4}} \qquad (36)$$

which is same as obtained by analytical approximation technique and zeroth order approximate solution is $u_{aat}(t) = a\cos\left(\sqrt{A_1 + \frac{3A_3 a^3}{4}}\right)t$ [20].

In figure 3 we have plotted the displacement x(t) from numerical solution for A_1 = 3.1415, A_3 = 0.15, a = 1 and compared the same obtained from Runge-Kutta (RK) and Analytical Approximation Technique (AAT) calculations. It is found that the displacement obtained from RK, ATHPM and AAT are matching very closely.

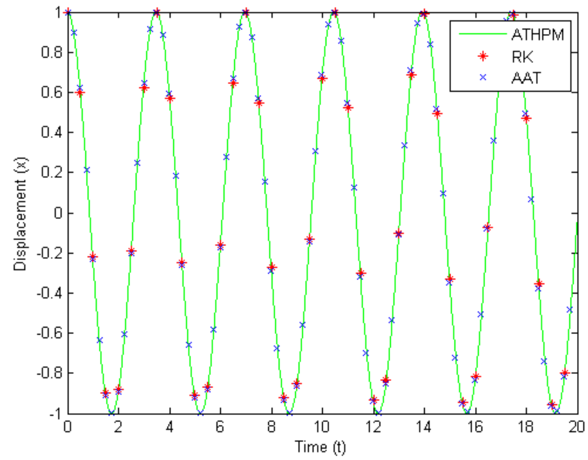

Figure 3. Time(t) vs displacement(x) curves obtained from numerical (RK), ATHMP and AAT with A_1 = 3.1415, A_3 = 0.15 and a = 1

Case 2

Here, we consider a straight Euler-Bernoulli beam that is subjected to an axial force of magnitude $\overline{P}$ as shown in figure 4 and figure 5. If we substitute $K = 0$, the equation of motion is directly obtained from (33) as

$$\frac{d^2u(t)}{dt^2} + A_1 u + A_3 u^3 = 0 \qquad (37)$$

where, $A_1 = a_1 + a_2 F, A_3 = a_3$ and a_1, a_2, a_3 are same as in (34).

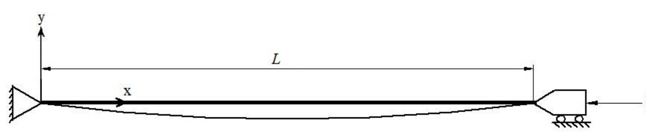

Figure 4. Schematic of Euler-Bernoulli simply supported beam subjected to an axial load

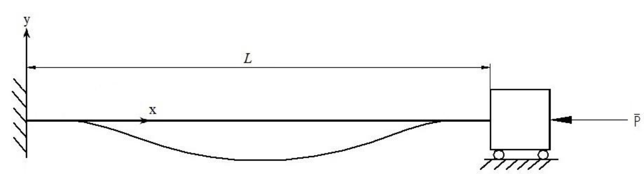

Figure 5. Schematic of Euler-Bernoulli clamped-clamped beam subjected to an axial load

Now the post-bucking load-deflection relation for the problem can be obtained from (37) as

$$F = -\frac{\left(a_1 + a_3 u^3\right)}{a_2} \qquad (38)$$

Neglecting the contribution of u in (38), the buckling load can be determined as

$$F_c = -\frac{a_1}{a_2} \qquad (39)$$

The center of the beam is subjected to the following initial conditions as $u(0)=a, u'(0)=0$. Here a is the non-dimensional maximum amplitude of oscillation. Comparing (20) and (33), we get the approximate solution up to the first order from (27) as

$$u(t) = \left(a - \frac{A_3 a^3}{32\omega^2}\right)\cos\omega t + \left(\frac{A_3 a^3}{32\omega^2}\right)\cos 3\omega t \qquad (40)$$

and the frequency of oscillation is obtained from (28) as

$$\omega = \sqrt{A_1 + \frac{3A_3 a^3}{4}} \qquad (41)$$

which is same as obtained by the max-min approach [21]. The approximate solution obtained by max-min approach is

$$u_{MMA}(t) = a\cos\left(\sqrt{A_1 + \frac{3A_3 a^2}{4}}\right)t \qquad (42)$$

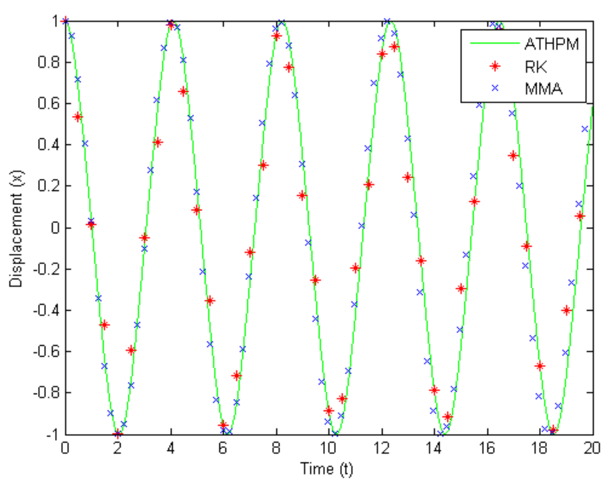

Figure 6. Time(t) vs displacement(x) curves obtained from numerical (RK), ATHMP and MMA with $A_1 = 1$, $A_2 = 0$, $A_3 = 1.814$ and a = 1

We have plotted the displacement x(t) from numerical solution for $A_1 = 1$, $A_2 = 0$, $A_3 = 1.814$, a = 1 and compared the same obtained from Runge-Kutta (RK) and Maximum-Minimum Approach (MMA) calculations. It is found that the displacement obtained from RK, ATHPM and MMA are matching very closely.

Case 3

A new kind of composite material known as functionally graded materials (FGMs) has received significant interest recently [22]. FGMs mare made from a mixture of materials and ceramics. They are further characterized by a smooth and continuous change of mechanical properties from one surface to another. FGMs were initially designed as thermal barrier materials for aerospace structures and fusion reactors where extremely higher temperature and large thermal gradient exist. With the increasing demand, FGMs have been widely used in general structures. Hence, many FGM structures have been studied, such as functionally graded(FG) beams, plates and shells. Due to the huge application of the beams in different fields such as civil, marine and aerospace engineering, it is necessary to study the dynamical behavior at large amplitudes which are effectively nonlinear.

Let us consider a straight functionally graded (FG) beam of length L, width b and thickness h which rests on an elastic nonlinear foundation and is subjected to an axial force of magnitude $\overline{P}$ as shown in figure 7.

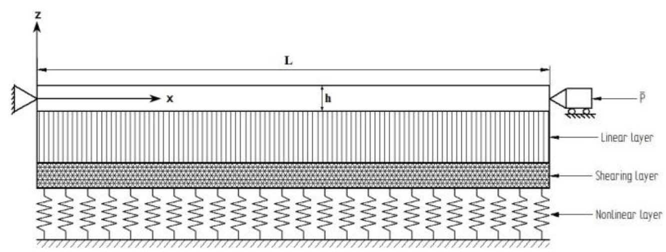

Figure 7. Schematic of FG beam with a nonlinear foundation

The beam is supported on an elastic function with cubic non-linearity and shearing layer. In this study, material properties are considered to vary in accordance with the rule of mixtures as

$$P = P_M V_M + P_C V_C \qquad (43)$$

where P and V are the material property and volume fraction, respectively, and the subscripts M and C refer to the metal and ceramic constituents respectively. Simple power law distribution from pure metal at the bottom face ($\bar{z} = -h/2$) to pure ceramic at the top face ($\bar{z} = +h/2$) in terms of volume fractions of the constituents is assumed as

$$V_C = \left(\frac{h + 2\bar{z}}{2h}\right)^n, V_M = 1 - V_C \qquad (44)$$

where n is the volume fraction exponent. The value of n equal to zero pre-presents a fully ceramic beam. The mechanical and thermal properties of FGMs are determined from the volume fraction of material constituents. We assume the non-homogeneous material properties such as the modulus of elasticity E, Poisson's ratio υ and mass density ρ which can be determined by substituting (44) into (43) as

$$E\left(\bar{z}\right)=E_M+\left(E_C-E_M\right)\left(\frac{h+2\bar{z}}{2h}\right)^n$$
$$\nu\left(\bar{z}\right)=\nu_M+\left(\nu_C-\nu_M\right)\left(\frac{h+2\bar{z}}{2h}\right)^n \qquad (45)$$
$$\rho\left(\bar{z}\right)=\rho_M+\left(\rho_C-\rho_M\right)\left(\frac{h+2\bar{z}}{2h}\right)^n$$

The force and moment results per unit length, based on classical theory of beams in a Cartesian coordinate system, can be written as

$$\begin{Bmatrix} N_{\bar{x}} \\ M_{\bar{x}} \end{Bmatrix}=b\begin{bmatrix} A_{11} & B_{11} \\ B_{11} & D_{11} \end{bmatrix}\begin{Bmatrix} \frac{\partial\bar{u}}{\partial\bar{x}}+\frac{1}{2}\left(\frac{\partial\bar{w}}{\partial\bar{x}}\right)^2 \\ \frac{\partial^2\bar{w}}{\partial\bar{x}^2} \end{Bmatrix} \qquad (46)$$

in which $\bar{w}$ and $\bar{u}$ are the transverse and axial displacements of the beam along the $\bar{z}$ and $\bar{x}$ directions respectively. The stiffness coefficients A_{11}, B_{11}, D_{11} are given as

$$\left(A_{11},B_{11},D_{11}\right)=\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{E(\bar{z})}{1-\nu^2(\bar{z})}\left(1,\bar{z},\bar{z}^2\right)dz \qquad (47)$$

After some mathematical simplifications, the governing equation of nonlinear free vibration of a FG beam in terms of transverse displacement can be written as

$$I\frac{\partial^2\bar{w}}{\partial\bar{t}^2}+b\left(D_{11}-\frac{B_{11}^2}{A_{11}}\right)\frac{\partial^4\bar{w}}{\partial\bar{x}^4}+\left(\bar{P}-\frac{bA_{11}}{2L}\int_0^L\left(\frac{\partial\bar{w}}{\partial\bar{x}}\right)^2 d\bar{x}\right.$$
$$\left.-\frac{bB_{11}}{2L}\left[\frac{\partial\bar{w}(L,\bar{t})}{\partial\bar{x}}-\frac{\partial\bar{w}(0,\bar{t})}{\partial\bar{x}}\right]\right)\frac{\partial^2\bar{w}}{\partial\bar{x}^2}=F_{\bar{w}} \qquad (48)$$

Here I and $F_{\bar{w}}$ are the inertial term and reaction of the elastic foundation on the beam which are defined as

$$I=\int_{-\frac{h}{2}}^{\frac{h}{2}}\rho\left(\bar{z}\right)d\bar{z},F_{\bar{w}}=-\bar{k}_L\bar{w}--\bar{k}_{NL}\bar{w}^3+\bar{k}_S\frac{\partial^2\bar{w}}{\partial\bar{x}^2} \qquad (49)$$

where, $\bar{k}_L$ and $\bar{k}_{NL}$ are linear and nonlinear elastic foundation coefficients, respectively, and $\bar{k}_S$ is the coefficient of shear stiffness of the elastic foundation. For convenience, we use the following non-dimensional variables

$$x=\frac{\bar{x}}{L},w=\frac{\bar{w}}{r},t=\bar{t}\sqrt{\frac{b\left(D_{11}-\frac{B_{11}^2}{A_{11}}\right)}{IL^4}} \qquad (50)$$

where, $r=\sqrt{I/A}$ is the radius of gyration of the cross-section. Using (48) and (49) together with the dimensionless variables defined in (50), the dimensional form of the governing equation becomes

$$I\frac{\partial^2 w}{\partial t^2}+b\left(D_{11}-\frac{B_{11}^2}{A_{11}}\right)\frac{\partial^4 w}{\partial x^4}+\left(P-\frac{1}{2}Q\int_0^L\left(\frac{\partial w}{\partial x}\right)^2 dx\right.$$
$$\left.-B\left[\frac{\partial w(1,t)}{\partial x}-\frac{\partial w(0,t)}{\partial x}\right]\right)\frac{\partial^2 w}{\partial x^2}+k_L w-k_{NL}w^3+k_S\frac{\partial^2 w}{\partial x^2}=0 \qquad (51)$$

where

$$P=\frac{\bar{P}L^2}{b\left(D_{11}-\frac{B_{11}^2}{A_{11}}\right)},Q=\frac{A_{11}r^2}{\left(D_{11}-\frac{B_{11}^2}{A_{11}}\right)},B=\frac{B_{11}r}{\left(D_{11}-\frac{B_{11}^2}{A_{11}}\right)}$$
$$k_L=\frac{\bar{k}_L L^4}{b\left(D_{11}-\frac{B_{11}^2}{A_{11}}\right)},k_{NL}=\frac{\bar{k}_{NL}L^4 r^2}{b\left(D_{11}-\frac{B_{11}^2}{A_{11}}\right)} \qquad (52)$$
$$k_S=\frac{\bar{k}_S L^2}{b\left(D_{11}-\frac{B_{11}^2}{A_{11}}\right)}$$

Assuming $w(x,t)=\psi(x)u(t)$ where $\psi(x)$ is the first eigen mode of the beam and applying the Galerkin method, the equation of motion is obtained as

$$\frac{d^2u}{dt^2}+A_1u+A_2u^2+A_3u^3=0 \qquad (53)$$

where, $A_1=a_1+a_PP+a_{k_L}+a_{k_S}, A_2=a_2, A_3=a_3+a_{k_{NL}}$

and $a_1,a_2,a_3,a_P,a_{k_L},a_{k_{NL}},a_{k_S}$ are given as

$$a_1 = \frac{\int_0^1 \frac{d^4\psi(x)}{dx^4}\psi(x)dx}{\int_0^1 \psi^2(x)dx}, a_P = \frac{\int_0^1 \frac{d^2\psi(x)}{dx^2}\psi(x)dx}{\int_0^1 \psi^2(x)dx}, a_{k_L} = k_L$$

$$a_{k_S} = -k_S a_P, a_2 = -B\left[\frac{d\psi(1)}{dx} - \frac{d\psi(0)}{dx}\right]a_P \quad (54)$$

$$a_3 = -Q\int_0^1 \psi^2(x)dx, a_{k_{NL}} = k_{NL}\frac{\int_0^1 \psi^4(x)dx}{\int_0^1 \psi^2(x)dx}$$

For simply supported beam,

$$\psi(x) = \sin\left(\frac{qx}{L}\right), q = \pi.$$

For clamped-clamped beam,

$$\psi(x) = \left(\cosh\frac{qx}{L} - \cos\frac{qx}{L}\right) - \frac{\cosh q - \cos q}{\sinh q - \sin q}\left(\sinh\frac{qx}{L} - \sin\frac{qx}{L}\right)$$

q=4.730041.

For clamped-simply supported beam,

$$\psi(x) = \left(\cosh\frac{qx}{L} - \cos\frac{qx}{L}\right) - \frac{\cosh q - \cos q}{\sinh q - \sin q}\left(\sinh\frac{qx}{L} - \sin\frac{qx}{L}\right)$$

q=3.926602.

Now the post-bucking load-deflection relation for the problem can be obtained from (13) as

$$F = -\left(a_1 + a_{k_L} + a_{k_S} + a_2 u + \left(a_3 + a_{k_{NL}}\right)u^2\right)/a_P \quad (55)$$

Neglecting the contribution of u in (55), the buckling load can be determined as

$$F_C = \frac{-\left(a_1 + a_{k_L} + a_{k_S}\right)}{a_P} \quad (56)$$

The center of the beam is subjected to the following initial conditions as $u(0) = a, u'(0) = 0$ where a denotes the non-dimensional maximum amplitude of oscillation. Comparing (20) and (33), we get the approximate solution up to the first order from (27) as

$$u_{ATHPM}(t) = -\frac{A_2 a^2}{2\omega^2} + \left(a + \frac{A_2 a^3}{3\omega^2} - \frac{A_3 a^3}{32\omega^2}\right)\cos\omega t + \frac{A_2 a^2}{6\omega^2}\cos 2\omega t + \frac{A_3 a^3}{32\omega^2}\cos 3\omega t \quad (57)$$

and the frequency of oscillation is obtained from (28) as

$$\omega = \sqrt{A_1 + \frac{3A_3 a^2}{4}} \quad (58)$$

which is same as calculated by variation iteration method [22].

Case 4

To develop the understanding of the nonlinear frequency of beam vibrations, this paper brings quintic nonlinearities into consideration. The analytical solution for geometrically nonlinear vibration of the Euler-Bernoulli beam including quintic nonlinearity is obtained using Aboodh transform and homotopy perturbation method [23]. The nonlinear ordinary differential equation of the beam vibration is extracted from the partial differential equation with first mode approximation based on the Galerkin theory.

Consider the Euler-Bernoulli beam of length l, mass per unit length μ, moment of inertia I and modulus of elasticity E which is axially compressed by loading P as shown in figure 8 and figure 9.

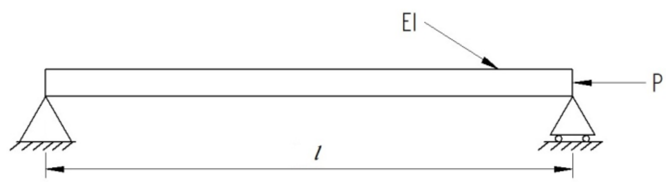

Figure 8. Schematic of uniform Euler-Bernoulli simply supported beam

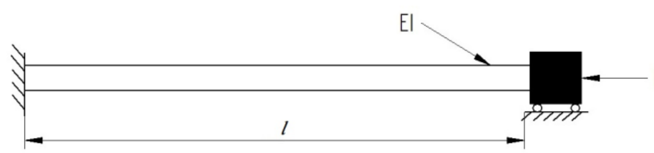

Figure 9. Schematic of uniform Euler-Bernoulli clamped-clamped beam

Denoting the transverse deflection by $w(x,\bar{t})$, the differential equation governing the equilibrium in the deformed situation is derived as

$$\frac{d^2}{dx^2}\left(\frac{EIw''(x,\bar{t})}{\sqrt{1+w'^2(x,\bar{t})}}\right)+Pw''(x,\bar{t})\left(1+\frac{3}{2}w'^2(x,\bar{t})\right) +\mu w''(x,\bar{t})=0 \quad (59)$$

Here, prime defines the derivative with respect to *x*. Using the approximation, we can write the exact expression for the curvature as

$$\frac{w''(x,\bar{t})}{\sqrt{\left(1+w'^2(x,\bar{t})\right)^3}}=w''(x,\bar{t})\left(1-\frac{3}{2}w'^2+\frac{15}{8}w'^4\right) \quad (60)$$

So, now we can rewrite (59) with the help of (60) as

$$EIw''''(x,\bar{t})\left(1-\frac{3}{2}w'^2+\frac{15}{8}w'^4\right)-9EIw'''w''w' +\frac{45}{2}EIw'''w''w'^3-3EIw''^3+\frac{45}{2}EIw''^3w'^2 +Pw''\left(1+\frac{3}{2}w'^2\right)+\mu w=0 \quad (61)$$

This equation is subjected to the following boundary conditions:

For simply supported (S-S) beam,

$w(0,\bar{t})=w''(0,\bar{t})=0$ and $w(l,\bar{t})=w''(l,\bar{t})=0$

For clamped-clamped (C-C) beam,

$w(0,\bar{t})=w(l,\bar{t})=0$ and $w'(0,\bar{t})=w'(l,\bar{t})=0$

Assuming $w(x,t)=\psi(x)u(t)$ where $\psi(x)$ is the first eigen mode of the beam vibration, it can be expressed as

For simply supported beam, $\psi(x)=\sin\left(\frac{\pi x}{l}\right)$

For clamped-clamped beam, $\psi(x)=\left(\frac{x}{l}\right)^2\left(1-\frac{x}{l}\right)^2$

Now, the Bubnov-Galerkin method gives

$$\int_0^l\left(EIw''''(x,\bar{t})\left(1-\frac{3}{2}w'^2+\frac{15}{8}w'^4\right)-9EIw'''w''w' +\frac{45}{2}EIw'''w''w'^3-3EIw''^3+\frac{45}{2}EIw''^3w'^2 +Pw''\left(1+\frac{3}{2}w'^2\right)+\mu w\right)\psi(x)dx=0 \quad (62)$$

Now, introducing the non-dimensional variables $u=\frac{\bar{u}}{l}$ and $t=\bar{t}\sqrt{EI/\mu L^4}$, the non-dimensional nonlinear equation of motion about its first bucking mode can be written as

$$\frac{d^2u}{dt^2}+A_1u+A_3u^3+A_5u^5=0 \quad (63)$$

where,

for S-S beam

$$A_1=\pi^4-\frac{Pl^2\pi^2}{EI},\ A_3=-\frac{3}{8}\pi^6-\frac{3}{8}\frac{Pl^2\pi^4}{EI},\ A_5=\frac{15\pi^8}{64}$$

for C-C beam,

$$A_1=500.534-\frac{12.142Pl^2}{EI},\ A_3=-6.654-\frac{0.1694Pl^2}{EI}$$
$$A_5=-0.3673$$

In this study, we have employed to solve the governing equation of vibration of quintic nonlinear beams by using ATHPM. With the initial conditions as $u(0)=a, u'(0)=0$ and comparing (20) and (33), we get the approximate solution up to first order from (27) as

$$u_{ATHPM}(t)=\left(a-\frac{A_3a^3}{32\omega^2}-\frac{A_5a^5}{384\omega^2}\right)\cos\omega t +\frac{1}{8\omega^2}\left(\frac{A_3a^3}{4}+\frac{5A_5a^5}{16}\right)\cos 3\omega t+\frac{A_5a^5}{384\omega^2}\cos 5\omega t \quad (64)$$

and the frequency of oscillation is obtained from (28) as

$$\omega=\sqrt{A_1+\frac{3A_3a^2}{4}+\frac{5A_5a^4}{8}} \quad (65)$$

which are same as calculated by maximum-minimum approach [21].

V. CONCLUSION

In this paper, the ATHPM has been implemented in order to analyse the equation of motion associated with the nonlinear vibration of Euler-Bernoulli beams. All the examples show that the presented results are in excellent agreement with those obtained by the exact numerical solution. We may also conclude that this technique is not only simple but also elegant way to study a wide class of realistic non-exactly solvable problems.

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